HOW TO USE LEVEL SET METHODS IN COMPUTER TOMOGRAPHY *

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1. Introduction

The level set methods, first proposed by Osher and Sethian, have been used in a rapidly growing number of areas, among others in computer tomography for the image reconstruction. I would like to introduce two image reconstruction tasks, i.e. image denoising and image deblurring by the level set methods.

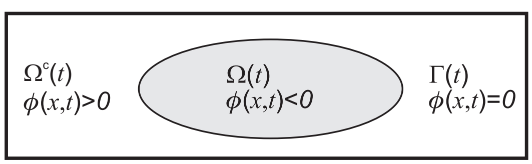
The main advantages of the level set methods are: topological changes in the evolving interface are easily handled, direct generalization to interface evolution in higher dimensions is possible, efficient numerical schemes are available that guarantee correct weak solutions satisfying the entropy condition, [2] p. 9–10.

2. Level Set Method

At first we derive the basic level set method. We need two objects for this type of the method: the level set function $\phi(\mathbf{x}, t)$ and a special set Γ called interface (generalizing from 2D to 3D is easy)

$$\phi(\mathbf{x}(t), t) : \mathbf{x}(t) \in \mathrm{R}^2, t \in [0, \infty[\to \mathrm{R}, \quad \Gamma(t) = \{\mathbf{x} \in \mathrm{R}^2 : \phi(\mathbf{x}, t) = 0\}.$$

The zero level set of function $\phi(\mathbf{x}, t)$ determines the interface Γ.



We start out from the equation

$$\phi(\mathbf{x}(t), t) = 0, \quad \forall \mathbf{x}(t) \in \Gamma(t), \tag{1}$$

and perform the differentiation of this formula with respect to the variable t, then we obtain

$$\frac{\mathrm{d}\phi(\mathbf{x}(t), t)}{\mathrm{d}t} = \phi_t + \nabla\phi \cdot \mathbf{U} = 0, \quad \mathbf{U} = \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}. \tag{2}$$

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If $\mathbf{U}$ has only the component pointing in the outward normal direction without the tangential component, then we can convert (2)

$$\phi_t + |\nabla\phi| U_n = 0, \quad \mathbf{U} = \mathbf{N}U_n + \mathbf{T}U_t, \quad \mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|}, \quad \nabla\phi \perp \mathbf{T}. \tag{3}$$

The initial condition $\phi_0(\mathbf{x}) = \phi(\mathbf{x}, 0)$ of (2) describes lay-out of the initial interface.

2.1. Level Set Dictionary

We will describe some useful notation in this section

- the normal direction vector $\mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|}$,
- the length of the interface $|\Gamma(t)| = \int |\nabla H(\phi(\mathbf{x}, t))| \, \mathrm{d}\mathbf{x} = \int \delta(\phi)|\nabla\phi| \, \mathrm{d}\mathbf{x}$,
- the volume of the interface $|\Omega(t)| = \int |H(\phi(\mathbf{x}, t))| \, \mathrm{d}\mathbf{x}$,
- the curvature of the interface $k = y$,
- the Heaviside function $H(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

3. Motion in the Normal Direction

We suppose that the front of the interface Γ is moving only in the normal direction. We start out from the basic level set equation (3) and put $U_n = -k = -\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$, $\mathrm{div} f(x, y) = f_x(x, y) + f_y(x, y)$ in this formula. We get a new level set equation

$$\phi_t = |\nabla\phi| \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right). \tag{4}$$

We deal with (3) and (4). The basic level set equation (3) is a hyperbolic equation. We can discretize this formula with upwind approach (see [1] p. 29–39) but (4) is a parabolic equation and we have to use another discretization approach. Instruction how to discretize this type of equation we can found in [1] p. 44. We can derive (4) from minimizing the length of the interface

$$\min_{\phi}\left(|\Gamma| = \int |\nabla H(\phi)| \, \mathrm{d}\mathbf{x}\right). \tag{5}$$

The level set equation (4) is commonly used in image reconstuction for denoising and deblurring of the image data.

4. Denoising and Deblurring

The recording of an image data usually involves degradation processes as are blurring or noise. The blurring (mostly deterministic) is caused by atmosphere turbulence, camera misfocus and relative movement or operator of Radon transform. The noise (mostly stochastic, Gauss white noise or Poisson noise) is caused by digital quantization, rain, lost packets by data transport, dust on lens.

Let u are clear image data (without any noise and blur), u_0 are damaged image data. Next we assume that the blurring is realized by a linear operator K and the random noise n has the zero mean value $E(n) = 0$ and variance σ^2. Now we can complete the linear model of image reconstruction. We have tree types:

- Deblurring task $u_0 = Ku$.
- Denoising task $u_0 = u + n$.
- Deblurring and denoising task $u_0 = Ku + n$.

We try to reconstruct (to find) clear data u from u_0. We introduce the functional

$$F_\alpha(u) = \frac{1}{2}\int (Ku - u_0)^2 \,\mathrm{d}\mathbf{x} + \alpha \int R(|\nabla u|)\,\mathrm{d}\mathbf{x}, \tag{6}$$

where $\alpha \in [0, \infty[$ is a weight of the regularization term and $R(|\nabla u|) = R\left(\sqrt{u_x^2 + u_y^2}\right) : \mathrm{R}^+ \to \mathrm{R}^+$ is the regularization function. If we set $\alpha = 0$ then we get $F_0(u) = \frac{1}{2}\int (Ku - u_0)^2 \,\mathrm{d}\mathbf{x}$. We try to minimize the mean value $\inf_u F_0(u)$, $||Ku - u_0||^2 = ||n||^2 \approx \sigma^2$. This task is indentical with $K^*Ku = K^*u_0$, where K^* is the adjoint operator to operator K. Unfortunately, we cannot perform the inversion of K^*K because the solution u does not depend continuously on the initial data. That is the reason why we need the regularization term $\alpha \neq 0$.

Functional $F_\alpha(u)$ has the form $\int G(x, y, u, u_x, u_y)\mathrm{d}x\mathrm{d}y$. The condition for the stationary point of this functional is $\delta F_\alpha(\hat{u}, h) = 0, \forall h$. It leads us to the Euler-Lagrange equation

$$\frac{\partial F_\alpha}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F_\alpha}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial F_\alpha}{\partial u_y} = 0. \tag{7}$$

Then we get

$$\begin{aligned} (K^*Ku - K^*u_0) - \alpha \mathrm{div}\left(\frac{R'(|\nabla u|)\nabla u}{|\nabla u|}\right) &= 0, \\ u_n &= 0. \end{aligned} \tag{8}$$

If we choose $R(\tau) = \tau$, then we obtain the equation

$$\begin{aligned} (K^*Ku - K^*u_0) - \alpha \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) &= 0, \\ u_n &= 0. \end{aligned} \tag{9}$$

If we choose $R(\tau) = \frac{\tau^2}{2}$, then we obtain the equation

$$\begin{aligned} (K^*Ku - K^*u_0) - \alpha\Delta u &= 0, \\ u_n &= 0. \end{aligned} \tag{10}$$

Computer tomography can be connected with deblurring method by the matrix K. Then the matrix K represents a projection matrix of direct discrete Radon transform. Deblurring of the image data means to do inverse discrete Radon transform.

5. Level Set Method for Denoising and Deblurring

We connect the level set method (equation) with the denoising and deblurring task. We start out from the initial value problem for the heat equation

$$\begin{aligned} u_t &= \Delta u, \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}). \end{aligned} \tag{11}$$

The solution of the heat equation (11) is given by

$$K(\mathbf{x}, \sigma) = \mathrm{e}^{-\frac{|\mathbf{x}|^2}{(4\sigma)}} (4\pi\sigma)^{-1}, \tag{12}$$

$$u(\mathbf{x}, \sigma) = K * u_0, \quad \sigma = t, \tag{13}$$

where $K * u_0$ denotes the linear convolution of K with u_0.

A standard operation on images is to convolve u_0 with a Gaussian kernel (12) of variance $\sigma > 0$ and it can give (13). This has the same effect as solving the initial value for the heat equation (11).

The first level set model for denoising and deblurring was introduced by Osher, Rudin, Fetami [3] p. 61,

$$\begin{aligned} u_t &= (K^*Ku - K^*u_0) - \alpha \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right), \\ u_n &= 0. \end{aligned} \tag{14}$$

This evolution procedure converges very slowly to its steady state since the parabolic term is singular for small gradients. The CFL condition for this equation has the form $\frac{\Delta t}{\Delta x^2} \leq c|\nabla u|$.

The second level set model was proposed by Marquina, Osher. They accelerated the movement of level curves of u and regularized the parabolic term in a nonlinear way. The equation has the form

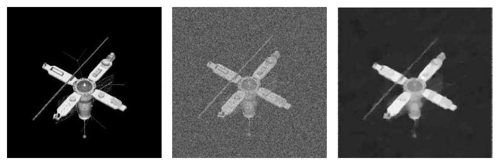
$$\begin{aligned} u_t &= |\nabla u|\left((K^*Ku - K^*u_0) - \alpha \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right), \\ u_n &= 0. \end{aligned} \tag{15}$$

The CFL condition for this equation has the form $\frac{\Delta t}{\Delta x^2} \leq c$.

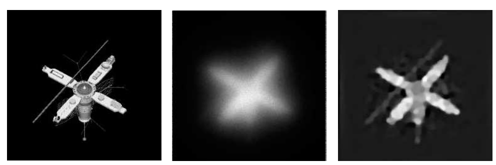
6. Numerical Experiments

We show three test tasks in this section. The first example consists of two images. The first image is blurred with 80 lost packets and the second one is its restoration.

The second example consists of three images and represents the denoising task. The first image is clear image u, the second one is the noisy image u_0 with Gauss white noise and the third one is its reconstruction.



The third example consists of three images and represents the deblurring task. The first image is clear image u, the second one is a very blurred image u_0 and the third one is its reconstruction.



7. Conclusion

We deal with a new way (called Level Set Method) in image processing. The level set methods are alternative to the classic methods, i.e. Fourier transform, Convolution, Wavelets, ...

Unfortunately, the objective termination rule of the iterative processes has not existed yet but the results of these methods are more than interesting.

References

[1] Osher, S. and Fedkiw, R.: *Level Set Methods and Dynamic Implicit Surfaces.* USA, Springer, ISBN 0-387-95482-1, 2003.

[2] Wei-Hsun, L.: *Mathematical Techniques in Object Matching and Computational Anatomy: a New Framework Based on the Level Set Method.* (Ph.D. thesis), UCLA report 03-23, June 2003.

[3] Bing, S.: *Topics in Variational PDE Image Segmentation, Inpainting and Denoising.* (Ph.D. Thesis), UCLA report 03-27, June 2003.